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A unified kinetic theory approach to external rarefied gas flows. Part 2. Application to a steady low-speed motion past a circular cylinder

By H. ATASSI
University of Notre Dame

AND S. F. SHEN
Cornell University

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The Navier–Stokes like equation derived by Atassi & Shen for low-speed rarefied flows is analysed for a circular geometry. Asymptotic solutions for both $Kn \ll 1$ and $Kn \gg 1$ are deduced using the method of matched asymptotic expansions. For $Kn \ll 1$, first- and second-order solutions are obtained and compared with the corresponding BGK treatment. For $Kn \gg 1$, this equation is used only to describe the far field, the inner solution being of the free-molecule type. The matching procedure leads to a *blockage correction* for the drag and heat-transfer rate. Results are compared with other theories and known experimental data.

1. Introduction

The theory we have developed in part 1 of this paper led to a single governing equation (part 1, equation (49)) for a low-speed rarefied gas flow over a nearly isothermal cylinder. Owing to the complicated nature of the right-hand side of this equation only numerical solutions appear to be possible for arbitrary Knudsen numbers. However, for the two asymptotic near-continuum and nearly free-molecule regimes, significant simplifications can be introduced into the governing equation and analytical solutions can be carried out for simple geometries. As an application we treat in this paper the problem of a steady low-speed motion past an isothermal infinite circular cylinder. The method of matched asymptotic expansions is used to deduce solutions to our governing equation for $Kn \ll 1$ and $Kn \gg 1$. The circular symmetry brings about further simplifications of the expressions of the rarefied terms. Using the same notation as in part 1, we set up a system of cylindrical polar co-ordinates centred on the axis of the cylinder (figure 1). The characteristic geometric length is taken to be the radius r_1 of the cylinder. For a given point M, the angles ϕ_{01} and ϕ_{02}, which define the two planes tangent to the cylinder, are symmetric with respect to the radial axis OM and equal to

$$\phi_0 = \arcsin r_1/r. \tag{1}$$

Accordingly, the functions G_{ijk} and L_{ijk} defined in appendix A, which appear in

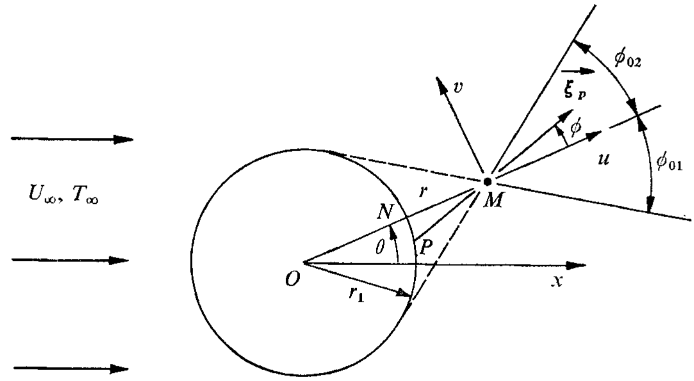

FIGURE 1. Geometrical configuration in the cross-sectional plane of the cylinder. $\boldsymbol{\xi}_p$ is the projection of the molecular velocity $\boldsymbol{\xi}$ and ϕ is the polar angle in the phase space. The projection of the distance travelled by the molecules from the cylinder is $PM = S_p$, the distance variable in the Knudsen layer is $NM = s$ and u and v are the components of the gas velocity in the polar co-ordinate system (r, θ).

the right-hand side of our governing equation, depend only on the radial distance r and the Knudsen number Kn. In addition, those with k an odd integer are equal to zero. The projection on the cross-sectional plane of the distance S travelled by the molecules from the surface of the cylinder is given by

$$S_p = r\cos\phi - (r_1^2 - r^2\sin^2\phi)^{\frac{1}{2}}. \tag{2}$$

Because of the importance of the stress tensor in our problem, we take the collision frequency

$$K = P_\infty/\mu_\infty. \tag{3}$$

Finally, we have the following governing equation:

$$\left[\nabla^2 - \frac{n_\infty}{\mu_\infty r}\left(\psi_\theta \frac{\partial}{\partial r} - \psi_r \frac{\partial}{\partial \theta}\right)\right]\nabla^2\psi = \frac{\mathscr{R}}{\mu_\infty r}, \tag{4}$$

with the boundary conditions

$$\left.\begin{aligned} &\psi_\theta = 0, \quad \psi_r - (Kn/\sqrt{\pi})\,(\psi_{rr} - \psi_r) = 0 \quad \text{at} \quad r = r_1, \\ &\psi = U_\infty r \sin\theta \quad \text{as} \quad r \to \infty. \end{aligned}\right\} \tag{5}$$

2. Near-continuum flow at low Reynolds number

For small Knudsen numbers, the rarefaction effects are significant only in a small layer of thickness $O(\lambda)$ in the immediate neighbourhood of the body. The main effect of this Knudsen layer is to modify the inner boundary conditions for the Navier–Stokes equation governing the rest of the field. However, in the case of a circular cylinder the only known explicit solution of the latter is the Stokes–Oseen solution, when the Reynolds number $Re \ll 1$. This problem was treated by Tamada & Yamamoto (1967) using the BGK equation. It is noted that the assumption of a small Reynolds number imposes the condition that

$$S_\infty = O(Kn\,Re)$$

and thus limits its practical usefulness. Nevertheless, it is particularly interesting here to show the applicability of our theory and the relative simplicity of our treatment.

2.1. *General form of the expansions*

The conditions of a small Knudsen number, $Kn = \lambda/r_1 \ll 1$, and a small Reynolds number, $Re = r_1/(\nu/U_\infty) \ll 1$, ν being the kinematic viscosity, imply that the viscous length ν/U_∞ is much larger than r_1, which is in turn much larger than the mean free path λ. This suggests that three regions of the flow field are to be considered: a Knudsen layer near the cylinder of size $O(\lambda)$ merging into a Stokes region of size $O(r_1)$, which in turn merges into an Oseen flow at a large distance of $O(\nu/U_\infty)$. We shall introduce appropriate variables and expand the governing equation (4) for each region. The solutions will be then matched using the method of matched asymptotic expansions.

The Knudsen layer. In the Knudsen layer we take the mean free path λ as unit length and define the distance variable $s^0 = (r-r_1)/\lambda$. The corresponding stream function is $\psi^0 = \psi/U_\infty \lambda$. We assume an expansion of the following form:

$$\psi^0 = \psi_0^0 + Kn\,\psi_1^0 + \dots. \tag{6}$$

Substituting (6) into (4) leads, after some reduction and double integration with respect to s^0, to

$$(1-4L_{522}^0)\frac{\partial^2\psi_0^0}{\partial s^{02}} + 4G_{412}^0\frac{\partial\psi_0^0}{\partial s^0} = a_1(\theta), \tag{7}$$

which expresses the fact that the total shearing stress $P_{r\theta}$ is constant, and

$$(1-4L_{522}^0)\frac{\partial^2\psi_1^0}{\partial s^{02}} + 4G_{412}^0\frac{\partial\psi_1^0}{\partial s^0} = a_2(\theta)s^0 + b_2(\theta)$$
$$+(1-4L_{522}^0)\frac{\partial\psi_0^0}{\partial s^0} + 4\left[L_{522}^1\frac{\partial^2\psi_0^0}{\partial s^{02}} - G_{412}^1\frac{\partial\psi_0^0}{\partial s^0}\right], \tag{8}$$

where a_1, a_2 and b_2 are functions of θ to be determined by the boundary conditions at the wall and matching with the Stokes expansion. The functions G_{ijk} and L_{ijk} are expanded in powers of Kn (see appendix A) to give

$$G_{ijk} = G_{ijk}^0 + Kn\,G_{ijk}^1 + o(Kn) \tag{9}$$

and a similar form for L_{ijk}.

Integration of (7) and (8) can be carried out and we get

$$\psi_0^0 = -v_0^*(0,\theta)\left[I_4(s^0) + \frac{2+\pi}{2\sqrt{\pi}}I_3(s^0)\right], \tag{10}$$

$$\frac{\partial\psi_1^0}{\partial s^0} = \exp[-F(s^0)]\left[a_2I_2(s^0) + b_2I_1(s^0) - v_1^*(0,\theta)\right.$$
$$\left.-v_0^*(0,\theta)s^0 - \frac{2+\pi}{2\sqrt{\pi}}v_0^*(0,\theta)I_5(s^0) - \frac{3}{2\sqrt{\pi}}I_6(s^0)\right], \tag{11}$$

where the normalized slip velocity $v^*(0,\theta) = -(\partial\psi^0/\partial s^0)_{s^0=0}$ has been expanded as

$$v^*(0,\theta) = v_0^*(0,\theta) + Kn\,v_1^*(0,\theta) + o(Kn), \tag{12}$$

and the functions $F(s^0), I_1(s^0), I_2(s^0), I_3(s^0), I_4(s^0), I_5(s^0)$ and $I_6(s^0)$ are given in appendix B. Expanding the boundary conditions (5) yields

$$\begin{aligned} v_0^*(0,\theta) &= \frac{1}{\sqrt{\pi}}\left(\frac{\partial v_0^*}{\partial s^0}\right)_{s^0=0}, \\ v_1^*(0,\theta)+\frac{v_0^*(0,\theta)}{\sqrt{\pi}} &= \frac{1}{\sqrt{\pi}}\left(\frac{\partial v_1^*}{\partial s^0}\right)_{s^0=0}. \end{aligned} \tag{13}$$

Application of conditions (13) to equation (8) gives

$$b_2(\theta) = -\frac{2+\pi}{2\sqrt{\pi}}v_1^*(0,\theta). \tag{14}$$

The remaining three unknown functions $v_0^*(0,\theta)$, $v_1^*(0,\theta)$ and $a_2(\theta)$ in (10) and (11) will be determined by matching these two equations with the Stokes solution for large values of s^0. The asymptotic expansions of ψ_0^0 and $\partial\psi_1^0/\partial s^0$ as $s^0 \to \infty$ are deduced from those of the various functions involved in their expressions, and are given in appendix B. We finally obtain

$$\psi_0^0 \sim v_0^*(0,\theta)\,[A_0 s^{02}+A_1 s^0+A_2]+\exp \quad \text{as} \quad s^0\to\infty, \tag{15}$$

$$\partial\psi_1^0/\partial s^0 \sim A_3 s^{02}+A_4 s^0+A_5+\exp \quad \text{as} \quad s^0\to\infty, \tag{16}$$

where 'exp' stands for terms that are exponentially small for large s^0, and

$$\left.\begin{aligned} A_0 &= -\frac{2+\pi}{4\sqrt{\pi}}, \quad A_1 = -\left[\frac{2+\pi}{2\sqrt{\pi}}A_1'+H_\infty\right], \\ A_2 &= -\left[\frac{2+\pi}{2\sqrt{\pi}}A_3'+A_4'H_\infty\right], \quad H_\infty = \lim_{s^0\to\infty}\exp[-F(s^0)], \\ A_3 &= \frac{a_2(\theta)}{2}-\frac{2+\pi}{4\sqrt{\pi}}v_0^*(0,\theta), \\ A_4 &= -\left[\frac{2+\pi}{2\sqrt{\pi}}v_1^*(0,\theta)+H_\infty v_0^*(0,\theta)+\frac{2+\pi}{2\sqrt{\pi}}A_1'v_0^*(0,\theta)\right], \\ A_5 &= -H_\infty v_1^*(0,\theta)+a_2(\theta)A_2'-\frac{2+\pi}{2\sqrt{\pi}}A_1'v_1^*(0,\theta) \\ &\quad -\frac{2+\pi}{2\sqrt{\pi}}A_5'v_0^*(0,\theta)-\frac{3}{2\sqrt{\pi}}A_6'. \end{aligned}\right\} \tag{17}$$

The quantities $A_1', A_2', A_3', A_4', A_5'$ and A_6' are given in appendix B.

The Stokes region. The unit length is the radius of the cylinder r_1 and we use the dimensionless distance variable $r^* = r/r_1$ and stream function

$$\psi^* = \psi/(r_1 U_\infty).$$

The functions G_{ijk} and L_{ijk} that appear as coefficients of the physical variables and their gradients in the expression for $\mathscr{R}$ in (4) have in common a factor of the order of $\exp[-(r^*-1)/Kn]$. For vanishing Kn and $(r^*-1) = O(1)$, this factor vanishes exponentially. It is then clear that the effect of rarefaction is confined to the thin Knudsen layer and will be felt in this region only through the inner boundary conditions. Equation (4) is then reduced to

$$\left[\nabla_*^2 - \frac{Re}{r^*}\left(\psi_\theta^* \frac{\partial}{\partial r^*} - \psi_{r^*}^* \frac{\partial}{\partial \theta}\right)\right] \nabla_*^2 \psi^* = O\left[\exp\left(-\frac{1}{Kn}\right)\right], \tag{18}$$

where ∇_* is the nabla operator corresponding to the distance variable r^*. For small values of the Reynolds number the leading term of (18) will be the biharmonic equation

$$\nabla_*^4 \psi^* = 0. \tag{19}$$

We now assume an expansion of the form

$$\psi^* = f_0(Re, Kn)\psi_0^*(r^*, \theta; Kn) + f_1(Re, Kn)\psi_1^*(r^*, \theta; Kn) + \dots, \tag{20}$$

where $f_{n+1}(Re, Kn)/f_n(Re, Kn) \to 0$ as $Re \to 0$. Expansion (20) should satisfy (18) and match with the Knudsen-layer solution near the wall and the Oseen solution at infinity.

The Oseen region. The unit length is the viscous length $\nu/U_\infty = r_1/Re$ and we introduce the new variables $\rho = r/(\nu/U_\infty)$ and $\Psi = \psi/\nu$. Equation (4) then becomes

$$\left[\nabla_\rho^2 - \frac{1}{\rho}\left(\Psi_\theta \frac{\partial}{\partial \rho} - \Psi_\rho \frac{\partial}{\partial \theta}\right)\right] \nabla_\rho^2 \Psi = O\left[\exp\left(-\frac{1}{Re\, Kn}\right)\right], \tag{21}$$

where ∇_ρ is the nabla operator corresponding to the distance variable ρ. We assume an expansion of the form

$$\Psi = F_0(Re, Kn)\Psi_0(\rho, \theta) + F_1(Re, Kn)\Psi_1(\rho, \theta) + \dots, \tag{22}$$

where $F_{n+1}(Re, Kn)/F_n(Re, Kn) \to 0$ as $Re \to 0$. This expansion should satisfy (21) and the free-stream condition

$$\Psi \to \rho \sin\theta \quad \text{as} \quad \rho \to \infty, \tag{23}$$

and match the Stokes solution (20). However, the circle is seen from this region as a singular point and therefore its effect far upstream should vanish as $\rho \to \infty$, whatever the values of Re and Kn may be. This leads immediately to taking

$$F_0(Re, Kn) \equiv 1, \tag{24}$$

$$\Psi_0(\rho, \theta) \equiv \rho \sin\theta, \tag{25}$$

$$\Psi_n(\rho, \theta) \to 0 \quad \text{as} \quad \rho \to \infty \quad \text{for all} \quad n \neq 0. \tag{26}$$

2.2. *The leading terms of the expansions*

We are now led to the classical problem of Proudman & Pearson (1957) with the difference that we have an additional parameter (Kn) and the boundary conditions at the wall are replaced by a matching condition with the Knudsen-layer solution. Therefore, we give in a succinct manner the matching procedure between the three solutions.

Upon substituting expansion (20) in (18) we get for the zero-order term

$$\nabla_*^4 \psi_0^* = 0. \tag{27}$$

The most general solution of this equation that is antisymmetric about $\theta = 0$ is

$$\psi_0^* = [A_0(Kn)r^* \log r^* + B_0(Kn)r^* + C_0(Kn)r^{*-1} + D_0(Kn)r^{*3}] \sin\theta$$
$$+ \sum_{m=2}^{\infty} B_m(Kn)r^{*m+2} + C_m(Kn)r^{*m} + D_m(Kn)r^{*2-m} + E_m(Kn)r^{*-m}] \sin m\theta. \tag{28}$$

Matching with the first term in the Oseen expansion (25), we get

$$\left.\begin{aligned} A_0(Kn) &= 1, \\ D_0(Kn) &= B_m(Kn) = C_m(Kn) = 0 \quad \text{for} \quad m \geqslant 2, \\ f_0(Re, Kn) &= [\log(1/Re)]^{-1}. \end{aligned}\right\} \tag{29}$$

However, it is convenient to take

$$f_0(Re, Kn) = [\log(1/Re) + C(Kn)]^{-1}, \tag{30}$$

where $C(Kn)$ is to be determined by matching conditions. The gauge function for the second term Ψ_1 in the Oseen expansion should then be

$$F_1(Re, Kn) = \frac{\phi_1(Kn)}{\log(1/Re) + C(Kn)}, \tag{31}$$

where $\phi_1(Kn)$ will also be determined by matching. The function Ψ_1 is a solution of the Oseen equation

$$(\nabla_\rho^2 - \partial/\partial\tilde{x})\nabla_\rho^2 \Psi_1 = 0, \tag{32}$$

where $\tilde{x} = \rho\cos\theta$. The general form of Ψ_1 is given by Tomotika & Aoi (1950). For small ρ, the Oseen expansion becomes

$$\Psi = \rho\sin\theta + F_1(Re, Kn)\,[\log\tfrac{1}{4}\rho + \gamma^* - 1]\rho\sin\theta + O(\rho^2\log\rho), \tag{33}$$

where $\gamma^* = 0{\cdot}5772\ldots$ is Euler's constant.

Matching (33) with the first term (28) of the Stokes solution gives

$$C(Kn) = B_0(Kn) + 1 - \gamma^* + \log 4, \quad \phi_1(Kn) = 1. \tag{34}$$

As a consequence of the nonlinearity of (21), we obtain

$$F_n(Re, Kn) = [\log(4/Re) + 1 - \gamma^* + B_0(Kn)]^{-n}, \tag{35}$$

$$f_n(Re, Kn) = [\log(4/Re) + 1 - \gamma^* + B_0(Kn)]^{-(n+1)}. \tag{36}$$

This leads to
$$\nabla_*^4 \psi_n^* = 0 \quad \text{for all} \quad n. \tag{37}$$

However, since (28) and (33) match perfectly as in the classical continuum case reported by Van Dyke (1964), we have the following form for the Stokes expansion:

$$\psi^* = f(Re, Kn)\psi_0^*, \tag{38}$$

with
$$f(Re, Kn) = f_0(Re, Kn) + \sum_{n=2}^{\infty} a_n f_n(Re, Kn), \tag{39}$$

where a_n are constants. The work of Kaplun (1957) can be easily extended here to show that $a_2 = -0{\cdot}87$.

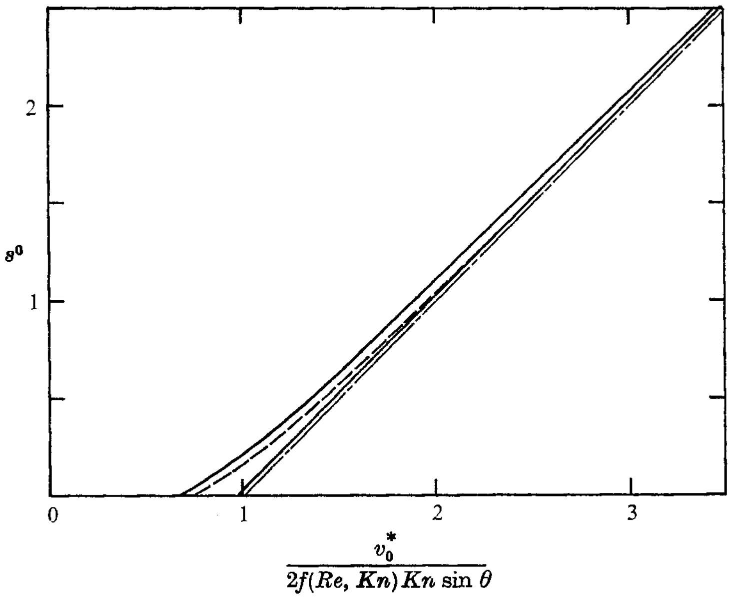

FIGURE 2. First-order velocity profile in the Knudsen layer. ——, profile and asymptote from the present theory. Results of Tamada & Yamamoto: – – –, profile; — - —, asymptote.

Now we determine B_0, C_0, D_0, D_m and E_m by matching the Stokes expansion (38) with the Knudsen-layer solutions (15) and (16). We assume that $B_0(Kn)$ can be expanded in power of Kn as

$$B_0(Kn) = B_{00} + B_{01} Kn + B_{02} Kn^2 \ldots, \tag{40}$$

and assume similar forms for the other functions. We then rewrite the Stokes expansion (38) in Knudsen variables. The matching procedure is straightforward and gives

$$\left.\begin{aligned} &D_m = E_m = 0 \quad \text{for all} \quad m, \\ &B_{00} = -\tfrac{1}{2}, B_{01} = \tfrac{1}{2}A_1/A_0 = 0\cdot 9869, \\ &B_{02} = \frac{1}{2}\frac{A_2 v_0^*(0,\theta) + A_5}{Kn f(Re, Kn)\sin\theta} = -4\cdot 335, \\ &C_{00} = \tfrac{1}{2}, \quad C_{01} = -\tfrac{1}{2}A_1/A_0 = -0\cdot 9869, \\ &C_{02} = \frac{1}{2}\frac{A_2 v_0^*(0,\theta) - A_5}{Kn f(Re, Kn)\sin\theta} = 4\cdot 008. \end{aligned}\right\} \tag{41}$$

The slip velocity at the wall is also determined:

$$\left.\begin{aligned} v_0^*(0,\theta) &= \frac{1}{A_0} f(Re, Kn) Kn \sin\theta = -1\cdot 3789 f(Re, Kn) Kn \sin\theta, \\ v_1^*(0,\theta) &= \frac{4\sqrt{\pi}}{2+\pi}\frac{A_1}{A_0} f(Re, Kn) Kn \sin\theta = 2\cdot 5838 f(Re, Kn) Kn \sin\theta. \end{aligned}\right\} \tag{42}$$

The profile of the first-order tangential velocity $v_0^*(s^0,\theta) = -\partial\psi_0^0/\partial s^0$ in the Knudsen layer is shown in figure 2. We have also plotted to first order

$$v^*(r^*, \theta) = -\partial\psi^*/\partial r^*,$$

which constitutes the asymptote to this profile and gives the conventional slip velocity

$$v_s^* = -(\partial\psi^*/\partial r^*)_{r^*=1} = -(1{\cdot}9738Kn - 8{\cdot}343Kn^2)f(Re, Kn)\sin\theta. \tag{43}$$

The drag of the cylinder is given by

$$D = 4\pi\mu U_\infty f(Re, Kn) \tag{44}$$

and the drag coefficient is

$$C_D = \frac{D}{n_\infty U_\infty^2 r_1} = \frac{4\pi}{Re}[f_0(Re, Kn) - 0{\cdot}87f_0^3(Re, Kn) + \ldots], \tag{45}$$

with $$f_0(Re, Kn) = [\log(4/Re) + \tfrac{1}{2} - \gamma^* + 0{\cdot}9869Kn - 4{\cdot}335Kn^2 + \ldots]^{-1}. \tag{46}$$

This result should be compared with that obtained by Tamada & Yamamoto using the linearized BGK equation. To first order, their value 1·016 is in a very close agreement with our 0·9869. To the second order, they obtained a much smaller value 0·0166 for the coefficient of Kn^2 compared with our value of $-4{\cdot}335$ in (46).

2.3. *Compressibility effects*

The pressure is easily calculated from the stream function ψ^*, with the results,

$$\tilde{P} = (P - P_\infty)/P_\infty = -2f(Re, Kn)\, Kn\, S_\infty \cos\theta/r^*. \tag{47}$$

From the equation of state of the gas, we obtain

$$\tilde{P} = \tilde{n} + \tilde{T}, \tag{48}$$

where $\tilde{n} = (n - n_\infty)/n_\infty$ and $\tilde{T} = (T - T_\infty)/T_\infty$. The order of $\tilde{T}$ was given in our general discussion of the compressibility effects in §4 of part 1 of this paper as the largest of S_∞^2 and $S_\infty Kn^2$. Therefore, from (47) and (48) we have

$$\tilde{n} = \tilde{P}. \tag{49}$$

3. Nearly free-molecule flow

For a large Knudsen number the general scheme of the flow can be simplified, making possible some analytical treatment of the problem. We consider three regions of the flow field. Near the body, there is a region of size $O(r_1)$ in which the number of collisions between molecules is relatively small. Accordingly, a free-molecule solution is the zero-order solution and can be directly analysed without using (4). This region merges at a distance $O(\lambda)$ into a region where the fluid obeys the governing equation (4) and which we shall call the *transition region*. Finally, noting that the assumption of a low-speed motion $S_\infty \ll 1$ leads to $\nu/U_\infty \gg \lambda$, we conclude that at a large distance $O(\nu/U_\infty)$ the rarefied part $\mathscr{R}$ of (4) vanishes exponentially as $\exp(-1/S_\infty)$ for small values of S_∞ to yield the usual Oseen region.

We introduce appropriate non-dimensional variables and stream functions as for the near-continuum flow. In the transition region the unit length is taken to be the mean free path λ and the following expansion is assumed:

$$\psi^0 = f_0(Re, Kn)\,\psi_0^0(r^0, \theta) + f_1(Re, Kn)\,\psi_1^0(r^0, \theta) + \ldots. \tag{50}$$

For the Oseen region the unit length is ν/U_∞ and, for similar reasons to those discussed for the near-continuum regime, we assume the following expansion:

$$\Psi = \rho \sin\theta + F_1(Re, Kn)\,\psi_1(\rho,\theta) + \dots. \tag{51}$$

The gauge functions f_1 and F_1 satisfy the conditions

$$f_1 \to 0, \quad F_1 \to 0 \quad \text{as} \quad Kn \to \infty \quad \text{and/or} \quad Re \to 0. \tag{52}$$

The function $\Psi_1(\rho,\theta)$ satisfies the Oseen equation (32). For a small Reynolds number an approximate form of the solution is given by Lagerstrom (1964) for the velocity field

$$\mathbf{u}^* = \mathbf{i} + 2\epsilon\mathbf{u}^{(1)} + O(Re^2), \tag{53}$$

where $\mathbf{i}$ is the unit vector parallel to the free-stream velocity, ϵ is a constant parameter and

$$\mathbf{u}^{(1)} = \nabla[\log\rho + \exp(\tfrac{1}{2}\rho\cos\theta)K_0(\tfrac{1}{2}\rho)] - \exp(\tfrac{1}{2}\rho\cos\theta)K_0(\tfrac{1}{2}\rho)\mathbf{i}. \tag{54}$$

K_0 is the modified Bessel function. The perturbation $2\epsilon\mathbf{u}^{(1)}$ to the uniform flow is due to the drag D of the cylinder acting as a singular force on the velocity field. The expression for the drag is

$$D = 2\epsilon\,.\,(2\pi\mu U_\infty). \tag{55}$$

Hence, the drag coefficient is

$$C_D = \frac{D}{\rho U_\infty^2 r_1} = \frac{4\pi}{Re}\epsilon. \tag{56}$$

On the other hand, the drag coefficient $C_{D,FM}$ of an isothermal circular cylinder in a free-molecule flow is well known (see Patterson 1956). For small S_∞ it can be well approximated by

$$C_{D,FM} = \pi^{\frac{1}{2}}(\tfrac{3}{2} + \tfrac{1}{4}\pi)/S_\infty. \tag{57}$$

This suggests that

$$F_1(Re, Kn) = \epsilon = \Delta(Kn)/Kn, \tag{58}$$

with $\Delta(Kn) = O(1)$. From (53) we get the expression for the stream function Ψ_1 for small ρ, which, when substituted in expansion (51), gives

$$\Psi = \rho\sin\theta + \epsilon[\log\tfrac{1}{4}\rho + \gamma^* - 1]\rho\sin\theta + O(\rho^2). \tag{59}$$

Expanding (59) in the transition region variables, we obtain

$$\psi^0 = r^0\sin\theta + \epsilon[\log(Re\,Kn) + \log\tfrac{1}{4}r^0 + \gamma^* - 1]r^0\sin\theta + O(Re^2\,Kn^2). \tag{60}$$

This immediately suggests taking

$$f_0(Re, Kn) = 1 + \epsilon\log(Re\,Kn), \quad f_1(Re, Kn) = \epsilon, \tag{61}$$

and also that for large r^0

$$\psi^0 \to r^0\sin\theta. \tag{62}$$

Now we return to study (4), which governs the transition region. First, we note that the radius of the circle in terms of the mean free path λ is $1/Kn$. For large Kn, the body is seen from this region as a singular point. Since the departure from the Maxwellian equilibrium is due to the presence of the body in the flow we should then expect its effect to be only a small perturbation to the main flow. This clearly suggests that

$$\psi_0^0 = r^0 \sin\theta, \tag{63}$$

which satisfies condition (62). To prove that (63) is also a solution to (4) we calculate the order of the rarefied part $\mathscr{R}$ of this equation. The angle $2\phi_0$ under which the cross-section of the cylinder is seen becomes small in this transition region. Expanding (1), we obtain

$$\phi_0 = 1/r^0 Kn + O(Kn^{-3}). \tag{64}$$

In a similar way we expand the functions G_{ijk} and L_{ijk}:

$$\left.\begin{aligned} G_{ijk} &\sim \frac{2}{(k+1)\pi Kn^{(k+1)} r^{0(k+1)}} J_i(r^0), \\ L_{ijk} &\sim \frac{2}{(k+1)\pi Kn^{(k+1)} r^{0(k+1)}} [J_i(r^0) + r^0 J_{i-1}(r^0)]. \end{aligned}\right\} \tag{65}$$

Consequently, the leading term of the rarefied part $\mathscr{R}$ is

$$\mathscr{R} \sim -\frac{1}{r}\frac{\partial^2}{\partial\theta\,\partial r}[r(P_{rr})_R], \tag{66}$$

since $(P_{rr})_R$ is the only component of the complementary stress tensor in which the functions G_{ijk} and L_{ijk} have the index $k = 0$. Rewriting (4) using the transition region variables yields

$$\left[\nabla_0^2 - \frac{Re\,Kn}{r^0}\left(\psi_\theta^0 \frac{\partial}{\partial r^0} - \psi_{r^0}^0 \frac{\partial}{\partial\theta}\right)\right]\nabla_0^2 \psi^0 = \frac{2}{r^{02}}\frac{\partial^2}{\partial\theta\,\partial r^0}[r^0 (P_{rr}^*)_R], \tag{67}$$

where ∇_0 is the nabla operator with λ as length unit, and

$$\begin{aligned}(P_{rr}^*)_R = {} & G_{320}\left(\frac{\tilde{n}_w - \tilde{n}^{(0)}}{S_\infty}\right) + (G_{520} - G_{320})\left(\frac{\tilde{T}_w - \tilde{T}}{S_\infty}\right) - 2G_{430} u^* \\ & + 2(L_{540} - L_{522})\frac{\partial u^*}{\partial r^0} - \frac{2}{5}\frac{\gamma}{\gamma - 1}\frac{1}{Pr\,S_\infty}(2L_{430} - L_{630})\frac{\partial \tilde{T}}{\partial r^0},\end{aligned} \tag{68}$$

where the subscript w denotes the conditions at the wall, γ is the specific heat ratio, Pr stands for the Prandtl number, $\tilde{n}^{(0)} = (n^{(0)} - n_\infty)/n_\infty$. $n^{(0)}$ is a partial density associated with the Chapman–Enskog velocity distribution function and is defined by equation (16) of part 1 of this paper. Here, it represents the density of the incoming molecules.

For an isothermal cylinder $\tilde{T}_w = 0$ and the variations of the density and temperature are of order S_∞ (see the discussion of compressibility effects in §4 of part 1). Therefore, the order of the right-hand side of (67) is that of the functions G_{ij0} and L_{ij0}, which is $1/Kn$. Upon substituting for ψ^0 by its expansion (50) in (67), and noting that $Re\,Kn = 2S_\infty$, we obtain for the first term

$$\nabla_0^4 \psi_0^0 = 0. \tag{69}$$

This confirms that (63) is the zero-order solution of (67). However, the first of (61) shows that there is a uniform slowing down of the flow velocity by an amount $\epsilon \log(Kn\,Re)$, owing to the drag of the cylinder seen as a singular force acting on the flow field. Before writing the equation for ψ_1^0, we note that from the free-molecule solution we have, at a large distance $O(\lambda)$,

$$\tilde{n}^{(0)} \sim S_\infty/Kn, \quad \tilde{T} \sim S_\infty/Kn \quad \text{and} \quad \tilde{n}_w = -\pi^{\frac{1}{2}} S_\infty \cos\theta.$$

The equation for ψ_1^0 follows after some arrangement and by neglecting

$$Re\,Kn = 2S_\infty;$$

$$\nabla_0^4 \psi_1^0 = \frac{4}{\pi\Delta(Kn) r^{02}} [\pi^{\frac{1}{2}} J_2(r^0) + 2J_3(r^0)] \sin\theta. \tag{70}$$

A direct integration gives the vorticity, which is, after matching with the Oseen solution (59),

$$\nabla_0^2 \psi_1^0 = \left[1 + \frac{\pi^{\frac{1}{2}} J_3(r^0) + 2J_4(r^0)}{\pi\Delta(Kn)}\right] \frac{2}{r^0} \sin\theta$$
$$- \frac{2}{\pi\Delta(Kn)} \left[\int_{r^0}^{\infty} \frac{\pi^{\frac{1}{2}} J_2(r^0) + 2J_3(r^0)}{r^{02}} dr^0\right] r^0 \sin\theta. \tag{71}$$

For small values of r^0, the leading term of the vorticity is

$$\nabla_0^2 \psi_1^0 \sim \left(1 + \frac{5}{4\pi^{\frac{1}{2}}\Delta(Kn)}\right) \frac{2}{r^0} \sin\theta. \tag{72}$$

From this last result we conclude that for small r^0 the expansion of ψ^0, including the first two terms ψ_0^0 and ψ_1^0, is given by

$$\psi^0 = (1 + \epsilon \log(Re\,Kn)) r^0 \sin\theta$$
$$+ \epsilon(1 + 5/4\pi^{\frac{1}{2}}\Delta(Kn)) r^0 \log r^0 \sin\theta + o(1/Kn). \tag{73}$$

Rewriting expansion (73) in the inner region variables yields

$$\psi^* = (1 - \epsilon_b) r^* \sin\theta + O(1/Kn), \tag{74}$$

where
$$\epsilon_b = \frac{1}{Kn}\left[\Delta(Kn) \log\frac{1}{Re} + \frac{5}{4\pi^{\frac{1}{2}}} \log Kn\right]. \tag{75}$$

Equation (74) says that the effective velocity of the flow as seen by the body in the free-molecule-like inner region is $U_\infty(1-\epsilon_b)$. As with our preliminary results (Atassi & Shen 1969), we call ϵ_b the *blockage factor*. However, we ought to point out that in those results, because we had not yet determined the second term ψ_1^0 of the transition region expansion, only the first term of ϵ_b was given. Finally, we conclude that the drag coefficient and heat-transfer rate can be calculated as for a free-molecule flow but with a free-stream velocity reduced by the blockage factor ϵ_b. This yields immediately

$$\frac{C_D}{C_{D,FM}} = \frac{Nu - Nu_0}{[Nu - Nu_0]_{FM}} = 1 - \epsilon_b, \tag{76}$$

where Nu is the Nusselt number and Nu_0 is the limit of Nu as $S_\infty \to 0$. The subscript FM denotes the corresponding free-molecule value with a free-stream velocity U_∞.

Combining (56), (57), (58), (75) and (76), we obtain the expression for ϵ:

$$\epsilon = \frac{6+\pi}{8\sqrt{\pi}} \frac{1}{Kn} \frac{1-(5/4\sqrt{\pi})\,[(\log Kn)/Kn]}{1+[(6+\pi)/8\sqrt{\pi}]\,[\log(1/Re)/Kn]}. \tag{77}$$

This leads immediately to

$$\frac{C_D}{C_{D,FM}} = \frac{1-5/4\sqrt{\pi}[(\log Kn)/Kn]}{1+[(6+\pi)/8\sqrt{\pi}]\,[\log(1/Re)/Kn]}. \tag{78}$$

It is interesting to note that the slowing down process of the velocity has increased from $\epsilon \log(Re\,Kn)$ at the outer boundary of the transition region to ϵ_b at its inner boundary. This blockage factor was not found by Liu & Passamaneck (1967), who used Lees' (1959) moment method. Also we note that ϵ_b contains two different effects. The first term of (75), $\epsilon \log(1/Re)$, is due to the action of the drag acting as a singular force on the far flow field, whereas its second term,

$$(5/4\sqrt{\pi})\,[(\log Kn)/Kn],$$

is due to collision effects in the transition region between the incoming molecules and those reflected from the body and thus depends on the kinetic theory model for the transition region. We therefore expect our blockage factor to depend on the average geometry of the body, and the form of (78) to hold for different geometries with changes only in the numerical factors. In fact, if we introduce in (78) the drag coefficient in the continuum regime,

$$C_{D,C} = \frac{4\pi}{Re \log(3{\cdot}703/Re)}, \tag{79}$$

and note that log 3·703 will introduce a term of $O(1/Kn)$ which is negligible to our order of approximation, we get

$$\frac{C_D}{C_{D,FM}} = \left(1+\frac{C_{D,FM}}{C_{D,C}}\right)^{-1}\left(1-\frac{5}{4\sqrt{\pi}}\frac{\log Kn}{Kn}\right). \tag{80}$$

If we retain only the first term of (80), we obtain Sherman's (1963) empirical formula

$$\frac{C_D}{C_{D,FM}} = \left(1+\frac{C_{D,FM}}{C_{D,C}}\right)^{-1}. \tag{81}$$

It should be pointed out that in our derivation of (78) the collision effects in the inner free-molecule-like region are not taken into account. Our blockage correction ϵ_b occurs before these effects, which would be of order $1/Kn$, are considered. Only those collision effects in the transition region are accounted for by the numerator of the right-hand side of (78). Further, ignoring them will lead to the simplified universal formula (81), which then can be derived in a very simple fashion by directly matching the Oseen solution (59) to the inner free-molecule solution.

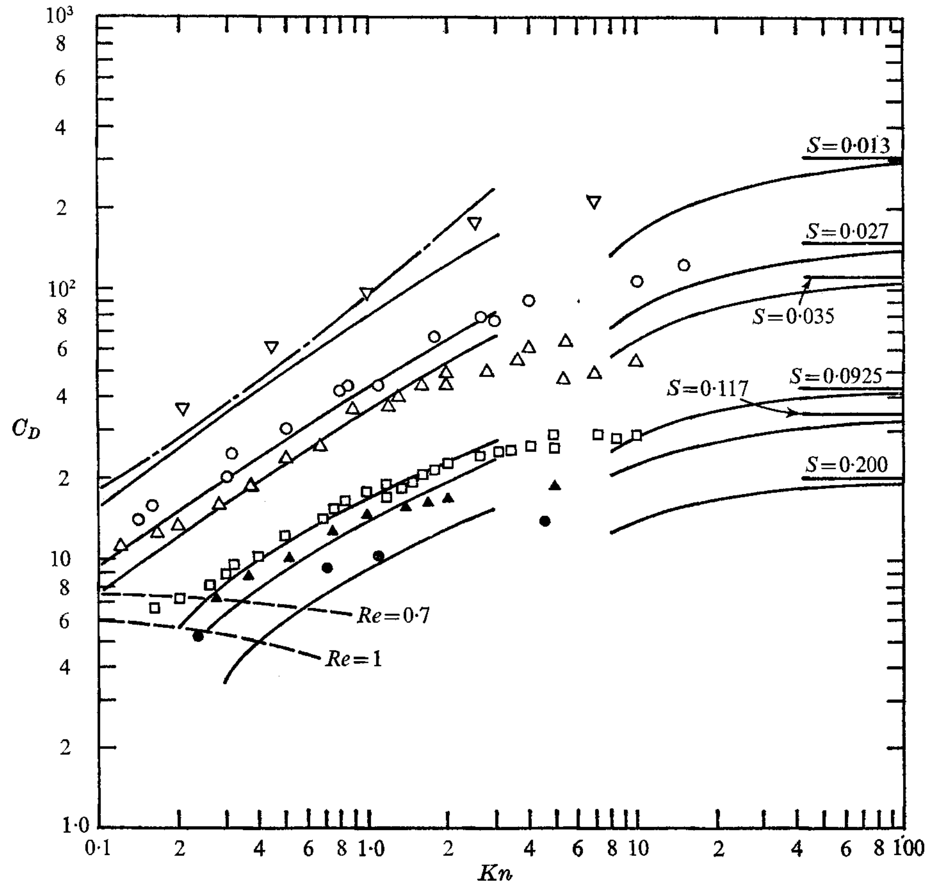

FIGURE 3. The drag of a cylinder versus the Knudsen number. Comparisons of the present theory with experimental data for different values of the molecular speed ratio S. Solid curves on the right- and left-hand side of the figure indicate nearly free-molecule and near-continuum theory respectively. ———, continuum theory for $S = 0{\cdot}013$. Data of Brun *et al.* (1963): △, $S = 0{\cdot}035$; □, $S = 0{\cdot}0925$; ▲, $S = 0{\cdot}117$. Data of Coudeville *et al.* (1965): ▽, $S = 0{\cdot}013$; ○, $S = 0{\cdot}027$; ●, $S = 0{\cdot}200$.

4. Comparisons with data

We have compared the results of (45) and (78) with the data of Brun, Fancy & Trotel (1963) and Coudeville, Trépau & Brun (1965). In the near-continuum regime, figure 3 shows a very good agreement of our theory with experiments. For this comparison we have dropped the term of $O(Kn^2)$ in (46) since, with its large coefficient, it limits the useful range of application of (45). We preferred to use two terms in the expansion of C_D as given by (45) because this gives a better agreement with experimental data for higher values of S_∞ and when the Reynolds number becomes closer to unity. Also, we note that an extension of (45) to a Knudsen number of order one still yields good agreement with experiments.

On the other hand, for nearly free-molecule flows we do not have enough experimental data in the range $Kn > 10$, because of the difficulties of drag

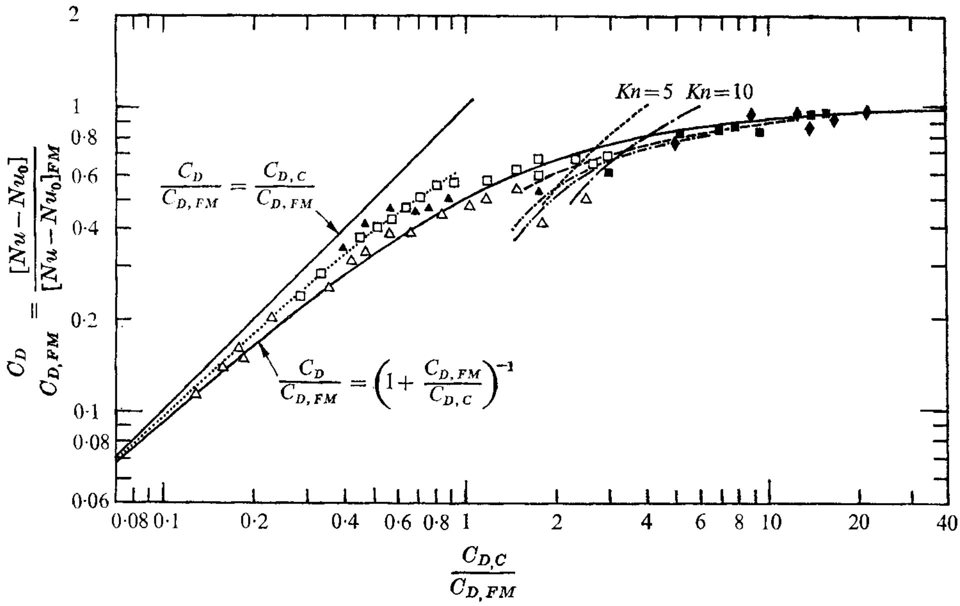

FIGURE 4. The drag and heat-transfer rate of a cylinder in Sherman's reduced variables. The asymptotic curves are plots of (78) for different values of the molecular speed ratio S. – – –, $S = 0{\cdot}34$; —·—, $S = 0{\cdot}12$; —··—, $S = 0{\cdot}035$; ·····, plot of (45) for $S = 0{\cdot}035$. Data of Brun *et al.* (1963): △, $S = 0{\cdot}035$; □, $S = 0{\cdot}0925$; ▲, $S = 0{\cdot}117$. Data of Atassi & Brun (1967): ■, $S = 0{\cdot}12$; ◆ $= 0{\cdot}34$.

measurements as the Knudsen number increases. However, for the two values $S_\infty = 0{\cdot}035$ and $S_\infty = 0{\cdot}0925$ experimental values seem to indicate agreement with curves. Also, we notice that the blockage factor is more important for smaller velocities. This fact can be explained by the following: as the mass motion of the gas becomes smaller with respect to molecular velocities, more of the effects of the perturbation produced by the cylinder on the far field can go upstream and the fluid feels these effects more.

Finally, using Sherman's reduced variables, we have compared on figure 4 our asymptotic solutions (45) and (78) with the drag measurements mentioned above and the heat-transfer data of Atassi & Brun (1967). The latter thus extend the available experimental data to the range of large Knudsen numbers. A very good agreement between data and theory is observed. Also, it is noted that this plot does not eliminate all Mach number effects; however, it makes the dependence on the Mach number rather weak. In the transition regime, experimental evidence shows that (81) is a satisfactory interpolation formula.

Appendix A. Expansion of G_{ijk} and L_{ijk} for $Kn \ll 1$

The functions G_{ijk} and L_{ijk} were defined in part 1 of this paper as

$$G_{ijk} = \frac{1}{\pi}\int_{\phi_{01}}^{\phi_{02}} \cos^j\phi \sin^k\phi J_i(S_p^0)\, d\phi, \tag{A 1}$$

$$L_{ijk} = G_{ijk} + \frac{1}{\pi}\int_{\phi_{01}}^{\phi_{02}} S_p^0 \cos^j\phi \sin^k\phi J_{i-1}(S_p^0)\, d\phi, \tag{A 2}$$

where ϕ is the phase angle shown on figure 1, $S_p^0 = S_p/\lambda$ and the $J_i(S_p^0)$ are the functions studied by Abramowitz (1953):

$$J_i(S_p^0) = \int_0^\infty X^i \exp - \left(X^2 + \frac{S_p^0}{X}\right) dX. \tag{A 3}$$

Because of the circular symmetry, $\phi_{02} = -\phi_{01} = \phi_0$, which was defined by (1). For $Kn \ll 1$, we have

$$\phi_0 = \tfrac{1}{2}\pi - (2\,Kn\, s^0)^{\frac{1}{2}} + O(Kn^{\frac{3}{2}}). \tag{A 4}$$

Expanding S_p (see equation (2)) in a similar way, we obtain, for $\phi < \frac{1}{2}\pi$,

$$S_p^0 = \frac{s^0}{\cos\phi} + \tfrac{1}{2}\,Kn\, s^{02}\frac{\tan^2\phi}{\cos\phi} + O(Kn^2). \tag{A 5}$$

This leads to

$$J_i(S_p^0) = J_i\left(\frac{s^0}{\cos\phi}\right) - \tfrac{1}{2}\,Kn\, s^{02}\left(\frac{\tan^2\phi}{\cos\phi}\right) J_{i-1}\left(\frac{s^0}{\cos\phi}\right) + O(Kn^2).$$

Because of the exponential decay of $J_i(s^0/\cos\phi)$ when $s^0/\cos\phi \to \infty$ ($\phi \to \frac{1}{2}\pi$), it can be shown that

$$\begin{aligned} G_{ijk}(s^0, Kn) = &\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} J_i\left(\frac{s^0}{\cos\phi}\right)\cos^j\phi \sin^k\phi\, d\phi \\ &- \tfrac{1}{2}\,Kn\, s^{02}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} J_{i-1}\left(\frac{s^0}{\cos\phi}\right)\cos^{j-3}\phi \sin^{k+2}\phi\, d\phi + o(Kn). \end{aligned} \tag{A 6}$$

The integrals in (A 6) can be evaluated by introducing the Cartesian co-ordinates $x = X\cos\phi$ and $y = X\sin\phi$, and splitting the double integrals into two factors. We finally get for $i = j+k+1$

$$G_{ijk} = 2J_k(0)J_j(s^0) - Kn\, s^{02}J_{k+2}(0)J_{j-3}(s^0) + o(Kn) \tag{A 7}$$

and for $i = j+k+3$

$$\begin{aligned} G_{ijk} = &\,2[J_k(0)\,J_{j+2}(s^0) + J_{k+2}(0)\,J_j(s^0)] \\ &- Kn\, s^{02}[J_{k+4}(0)J_{j-3}(s^0) + J_{k+2}(0)J_{j-1}(s^0)] + o(Kn). \end{aligned} \tag{A 8}$$

Equations (A 7) and (A 8) can be summarized as

$$G_{ijk} = G_{ijk}^0 + Kn\, G_{ijk}^1 + o(Kn),$$

where the definitions of G_{ijk}^0 and G_{ijk}^1 are quite obvious.

In a similar way we obtain the expansion of L_{ijk}:

$$L_{ijk} = L^0_{ijk} + Kn\, L^1_{ijk} + o(Kn), \tag{A 9}$$

where

$$\left.\begin{aligned} L^0_{ijk} &= G^0_{ijk} + s^0 G^0_{i-1,j-1,k}, \\ L^1_{ijk} &= -\tfrac{1}{2} s^{03} G^0_{i-2,j-4,k+2}. \end{aligned}\right\} \tag{A 10}$$

Appendix B. The functions of $F(s^0)$ and $I_i(s^0)$ of the Knudsen-layer solutions

$$F(s^0) = \frac{2}{\sqrt{\pi}} \int_0^{s^0} \frac{J_1(t)}{1-(2/\sqrt{\pi})[J_2(t)+tJ_1(t)]}\, dt,$$

$$H_\infty = \lim_{s^0\to\infty} \exp[-F(s^0)] = 0\cdot4420\ldots.$$

$$I_1(s^0) = \int_0^{s^0} \frac{\exp[F(t)]}{1-(2/\sqrt{\pi})[J_2(t)+tJ_1(t)]}\, dt,$$

$$A'_1 = \lim_{s^0\to\infty} \{I_1(s^0)\exp[-F(s^0)] - s^0\} = 0\cdot6322\ldots.$$

$$I_2(s^0) = \int_0^{s^0} \frac{t\exp[F(t)]}{1-(2/\sqrt{\pi})[J_2(t)+tJ_1(t)]}\, dt,$$

$$A'_2 = \lim_{s^0\to\infty} \{I_2(s^0)\exp[-F(s^0)] - \tfrac{1}{2}s^{02}\} = 1\cdot6484\ldots.$$

$$I_3(s^0) = \int_0^{s^0} I_1(t)\exp[-F(t)]\, dt,$$

$$A'_3 = \lim_{s^0\to\infty} [I_3(s^0) - \tfrac{1}{2}s^{02} - A'_1 s^0] = -0\cdot6039\ldots.$$

$$I_4(s^0) = \int_0^{s^0} \exp[-F(t)]\, dt,$$

$$A'_4 = \lim_{s^0\to\infty} [I_4(s^0)/H_\infty - s^0] = 0\cdot9140\ldots.$$

$$I_5(s^0) = \int_0^{s^0} I_1(t)\, dt,$$

$$A'_5 = \lim_{s^0\to\infty} [I_5(s^0)H_\infty - \tfrac{1}{2}s^{02} - A'_1 s^0] = -1\cdot6450\ldots.$$

$$I_6 = \int_0^{s^0} \frac{t^2 J_{-2}(t)\,[t\,\partial^2\psi^0_0/\partial t^2 - \partial\psi^0_0/\partial t]\exp[F(t)]}{1-(2/\sqrt{\pi})[J_2(t)+tJ_1(t)]}\, dt,$$

$$A'_6 = \lim_{s^0\to\infty} I_6(s^0) = 0\cdot345\ldots.$$

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